

# GLOBAL UNITS MODULO ELLIPTIC UNITS AND 2-IDEAL CLASS GROUPS

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## Abstract

Let  $p \in \{2, 3\}$ , and let  $k$  be an imaginary quadratic number field in which  $p$  decomposes into two distinct primes  $\mathfrak{p}$  and  $\bar{\mathfrak{p}}$ . Let  $k_\infty$  be the unique  $\mathbb{Z}_p$ -extension of  $k$  which is unramified outside of  $\mathfrak{p}$ , and let  $K_\infty$  be a finite extension of  $k_\infty$ , abelian over  $k$ . We prove that in  $K_\infty$ , the projective limit of the  $p$ -class group and the projective limit of units modulo elliptic units share the same  $\mu$ -invariant and the same  $\lambda$ -invariant. Then we prove that up to a constant, the characteristic ideal of the projective limit of the  $p$ -class group coincides with the characteristic ideal of the projective limit of units modulo elliptic units.

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**Key words:** Elliptic units, Iwasawa theory.

## 1 Introduction.

Let  $p$  be a prime number, and let  $k$  be an imaginary quadratic number field in which  $p$  decomposes into two distinct primes  $\mathfrak{p}$  and  $\bar{\mathfrak{p}}$ . Let  $k_\infty$  be the unique  $\mathbb{Z}_p$ -extension of  $k$  which is unramified outside of  $\mathfrak{p}$ , and let  $K_\infty$  be a finite extension of  $k_\infty$ , abelian over  $k$ . Let  $G_\infty$  be the Galois group of  $K_\infty/k$ . We choose a decomposition of  $G_\infty$  as a direct sum of a finite group  $G$  (the torsion subgroup of  $G_\infty$ ) and a topological group  $\Gamma$  isomorphic to  $\mathbb{Z}_p$ ,  $G_\infty = G \times \Gamma$ . For any  $n \in \mathbb{N}$ , let  $K_n$  be the field fixed by  $\Gamma_n := \Gamma^{p^n}$ , and let  $G_n := \text{Gal}(K_n/k)$ . Remark that there may be different choices for  $\Gamma$ , but when  $p^n$  is larger than the order of the  $p$ -part of  $G$ , the group  $\Gamma_n$  does not depend on the choice of  $\Gamma$ .

Let  $F/k$  be an abelian extension of  $k$ . If  $[F : k] < \infty$ , we denote by  $\mathcal{O}_F$  the ring of integers of  $F$ . Then we write  $\mathcal{O}_F^\times$  for the group of global units of  $F$ , and  $C_F$  for the group of elliptic units of  $F$  (see section 2). Also we let  $A_F$  be the  $p$ -part of the class group  $\text{Cl}(\mathcal{O}_F)$  of  $\mathcal{O}_F$ . We set  $\mathcal{E}_F := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F^\times$  and  $\mathcal{C}_F := \mathbb{Z}_p \otimes_{\mathbb{Z}} C_F$ . When  $F/k$  is infinite, we define  $\mathcal{E}_F$ ,  $\mathcal{C}_F$  and  $A_F$ , by taking projective limits over finite sub-extensions, under the norm maps. For any  $n \in \mathbb{N} \cup \{\infty\}$ , we set  $\mathcal{E}_n := \mathcal{E}_{K_n}$ ,  $\mathcal{C}_n := \mathcal{C}_{K_n}$ , and  $A_n := A_{K_n}$ .

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For any profinite group  $\mathcal{G}$ , and any commutative ring  $R$ , we define the Iwasawa algebra

$$R[[\mathcal{G}]] := \varprojlim R[\mathcal{H}],$$

where the projective limit is over all finite quotient  $\mathcal{H}$  of  $\mathcal{G}$ . In case  $\mathcal{G} = G_\infty$  (respectively  $\mathcal{G} = \Gamma$ ), we shall write

$$\Lambda := \mathbb{Z}_p[[G_\infty]] \quad (\text{respectively} \quad \Lambda' := \mathbb{Z}_p[[\Gamma]]).$$

Then  $A_\infty$  and  $\mathcal{E}_\infty/\mathcal{C}_\infty$  are naturally  $\Lambda$ -modules. As we shall see below, they are finitely generated and torsion over  $\Lambda'$ . Let us fix a topological generator  $\gamma$  of  $\Gamma$ , and set  $T := \gamma - 1$ . Then for any finite extension  $L/\mathbb{Q}_p$ ,  $\mathcal{O}_L[[\Gamma]]$  is isomorphic to  $\mathcal{O}_L[[T]]$ , where  $\mathcal{O}_L$  is the ring of integers of  $L$ . It is well known that  $\mathcal{O}_L[[T]]$  is a noetherian, regular, local domain. We also recall that  $\mathcal{O}_L[[T]]$  is a unique factorization domain. If  $u$  is a uniformizer of  $\mathcal{O}_L$ , then the maximal ideal  $\mathfrak{m}$  of  $\mathcal{O}_L$  is generated by  $u$  and  $T$ , and  $\mathcal{O}_L[[T]]$  is a compact topological ring with respect to its  $\mathfrak{m}$ -adic topology. A morphism  $f : M \rightarrow N$  between two finitely generated  $\mathcal{O}_L[[T]]$ -module is called a pseudo-isomorphism if its kernel and its cokernel are finite. If a finitely generated  $\mathcal{O}_L[[T]]$ -module  $M$  is given, then one may find elements  $P_1, \dots, P_r$  in  $\mathcal{O}_L[T]$ , irreducible in  $\mathcal{O}_L[[T]]$ , and nonnegative integers  $n_0, \dots, n_r$ , such that there is a pseudo-isomorphism

$$M \longrightarrow \mathcal{O}_L[[T]]^{n_0} \oplus \bigoplus_{i=1}^r \mathcal{O}_L[[T]]/(P_i^{n_i}).$$

Moreover, the integer  $n_0$  and the ideals  $(P_1^{n_1}), \dots, (P_r^{n_r})$ , are uniquely determined by  $M$ . If  $n_0 = 0$ , then the ideal generated by  $P_1^{n_1} \cdots P_r^{n_r}$  is called the characteristic ideal of  $M$ , and is denoted by  $\text{char}_{\mathcal{O}_L[[T]]}(M)$ .

We denote by  $\mathbb{C}_p$  a completion of an algebraic closure of  $\mathbb{Q}_p$ . Let  $\chi : G \rightarrow \mathbb{C}_p^\times$  be an irreducible character of  $G$ . Let  $\mathbb{Q}_p(\chi) \subset \mathbb{C}_p$  be the abelian extension of  $\mathbb{Q}_p$  generated by the values of  $\chi$ . We denote by  $\mathbb{Z}_p(\chi)$  the ring of integers of  $\mathbb{Q}_p(\chi)$ . The group  $G$  acts naturally on  $\mathbb{Q}_p(\chi)$  if we set, for all  $g \in G$  and all  $x \in \mathbb{Q}_p(\chi)$ ,  $g.x := \chi(g)x$ . For any  $\mathbb{Z}_p[G]$ -module  $Y$ , we define the  $\chi$ -quotient  $Y_\chi$  by  $Y_\chi := \mathbb{Z}_p(\chi) \otimes_{\mathbb{Z}_p[G]} Y$ . If  $Y$  is a  $\Lambda$ -module, then  $Y_\chi$  is a  $\mathbb{Z}_p(\chi)[[T]]$ -module in a natural way. As a particular case,  $\Lambda_\chi \simeq \mathbb{Z}_p(\chi)[[T]]$ . For any finitely generated  $\Lambda_\chi$ -module  $Z$ , we shall denote  $\text{char}_{\Lambda_\chi} Z$  simply by  $\text{char}_\chi Z$ .

The goal of this article is to prove Theorem 1.1 below, which is a raw formulation of the (one-variable) main conjecture at  $p = 2$  or  $p = 3$ . In [13, Theorem 4.1] and [14, Theorem 2], Rubin used Euler systems to prove the main conjectures for  $\mathbb{Z}_p$  or  $\mathbb{Z}_p^2$  extensions of a finite abelian extension  $F$  of  $k$ , where  $p \nmid w_k[F : k]$ ,  $w_k$  being the number of roots of unity in  $k$ . Inspired by the ideas of Rubin, Greither used Euler systems to prove the main conjecture for cyclotomic units and for the cyclotomic  $\mathbb{Z}_p$ -extension  $F_\infty/F$ , with  $F_\infty$  abelian over  $\mathbb{Q}$  (see [5, Theorem 3.2]). Bley proved the main conjecture when  $p \nmid 2\#(\text{Cl}(\mathcal{O}_k))$ , and when there is a nonzero ideal  $\mathfrak{f}$  of  $\mathcal{O}_k$ , prime to  $\mathfrak{p}$ , such that for all  $n \in \mathbb{N}$ ,  $K_n = k(\mathfrak{f}\mathfrak{p}^n)$  is the ray class field of  $k$  modulo  $\mathfrak{f}\mathfrak{p}^n$  (see [2, Theorem 3.1]). More recently, Hassan Oukhaba adapted Rubin's method and obtained the divisibility relation

$$\text{char}_\chi(A_\infty) | p^{m_\chi} \text{char}_\chi(\mathcal{E}_\infty/\mathcal{C}_\infty) \quad \text{for some } m_\chi \in \mathbb{N}, \quad (1.1)$$

for  $p = 2$ , still under the condition  $2 \nmid [K_0 : k]$  (see [8]). In [17], we proved the main conjecture for the extension  $K_\infty/K$  when  $p \notin \{2, 3\}$ . When  $p \in \{2, 3\}$ , we obtained the divisibility relation (1.1). Here we prove that an equality holds also for  $p \in \{2, 3\}$ , up to a constant in  $\mathbb{Z}_p(\chi)$ .

**Theorem 1.1** *Let  $p \in \{2, 3\}$ , and let  $\chi$  be an irreducible  $\mathbb{C}_p$  character on  $G$ . There are  $a_\chi \in \mathbb{N}$  and  $b_\chi \in \mathbb{N}$  such that*

$$\mathbf{u}_\chi^{a_\chi} \text{char}_\chi(A_{\infty, \chi}) = \mathbf{u}_\chi^{b_\chi} \text{char}_\chi(\mathcal{E}_\infty / \mathcal{C}_\infty)_\chi, \quad (1.2)$$

where  $\mathbf{u}_\chi$  is a uniformizer of  $\mathbb{Z}_p(\chi)$ .

Let  $L/\mathbb{Q}_p$  be a finite algebraic extension. For any finitely generated torsion  $\mathcal{O}_L[[T]]$ -module  $M$ , we recall that the  $\lambda$ -invariant  $\lambda(M)$  of  $M$  is the  $\mathcal{O}_L$ -rank of  $M$ , which is also the Weierstrass degree of any generator of  $\text{char}_{\mathcal{O}_L[[T]]} M$ . Let  $\mathbf{u}_L$  be a uniformizer of  $\mathcal{O}_L$ . Then the  $\mu$ -invariant  $\mu(M)$  of  $M$  is the maximal power of  $\mathbf{u}_L$  which divides  $\text{char}_{\mathcal{O}_L[[T]]} M$ .

For any finitely generated  $\Lambda$ -module  $M$ , we recall that

$$\sum_\chi \lambda(M_\chi) = \lambda(M), \quad (1.3)$$

where the sum is over all irreducible  $\mathbb{C}_p$  characters of  $G$ . In particular, (1.3) and the divisibility relation (1.1) gives us

$$\lambda(A_\infty) = \sum_\chi \lambda(A_{\infty, \chi}) \leq \sum_\chi \lambda(\mathcal{E}_{\infty, \chi} / \mathcal{C}_{\infty, \chi}) = \lambda(\mathcal{E}_\infty / \mathcal{C}_\infty), \quad (1.4)$$

where the sums are over all irreducible  $\mathbb{C}_p$  characters of  $G$ . Hence in order to prove Theorem 1.1, it suffices to show that

$$\lambda(\mathcal{E}_\infty / \mathcal{C}_\infty) = \lambda(A_\infty). \quad (1.5)$$

Indeed from (1.5) and (1.4) we deduce  $\lambda(A_{\infty, \chi}) = \lambda(\mathcal{E}_{\infty, \chi} / \mathcal{C}_{\infty, \chi})$  for any  $\mathbb{C}_p$  character  $\chi$  of  $G$ , and this last equality together with (1.1) induce Theorem 1.1. For  $p \notin \{2, 3\}$ , a proof of (1.5) is sketched in [3, III.2.1]. A complete proof is given in the cyclotomic case by J.R. Belliard in [1]. In this article, we prove the elliptic version of [1, Theorem 6.2]. More precisely we show the two following facts (see Theorem 5.1).

(i) The  $\Lambda'$ -modules  $\mathcal{E}_\infty / \mathcal{C}_\infty$  and  $A_\infty$  share the same  $\lambda$ -invariant and the same  $\mu$ -invariant.

(ii) The  $\Lambda'$ -modules  $\mathcal{U}_\infty / \mathcal{C}_\infty$  and  $B_\infty$  share the same  $\lambda$ -invariant and the same  $\mu$ -invariant, where  $\mathcal{U}_\infty$  is a module of semi-local units and  $B_\infty$  is the Galois group of the maximal abelian pro- $p$ -extension of  $K_\infty$  which is unramified outside the primes above  $\mathfrak{p}$  (see Section 3).

As mentioned above, from (i) and (1.1) one can derive Theorem 1.1. We will closely follow the ideas of Belliard. Although we are mostly interested in the case  $p = 2$  or  $p = 3$ , the method works for any prime number  $p$ .

## 2 Elliptic units.

For  $L$  and  $L'$  two  $\mathbb{Z}$ -lattices of  $\mathbb{C}$  such that  $L \subseteq L'$  and  $[L' : L]$  is prime to 6, we denote by  $z \mapsto \psi(z; L, L')$  the elliptic function defined in [11]. For  $\mathfrak{m}$  a nonzero proper ideal of  $\mathcal{O}_k$ , and  $\mathfrak{a}$  a nonzero ideal of  $\mathcal{O}_k$  prime to  $6\mathfrak{m}$ , G. Robert proved that  $\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}) \in k(\mathfrak{m})$ , where  $k(\mathfrak{m})$  is the ray class field of  $k$ , modulo  $\mathfrak{m}$ . Let  $S(\mathfrak{m})$  be the set of maximal ideals of  $\mathcal{O}_k$  which divide  $\mathfrak{m}$ . Then  $\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})$  is a unit if and only if  $|S(\mathfrak{m})| = 1$ . More precisely, if we denote by  $w_{\mathfrak{m}}$  the number of roots of unity of  $k$  which are congruent to 1

modulo  $\mathfrak{m}$ , and if we write  $w_k$  for the number of roots of unity of  $k$ , then by [10, Corollaire 1.3, (iv)], we have

$$\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}) \mathcal{O}_{k(\mathfrak{m})} = \begin{cases} (1) & \text{if } 2 \leq |S(\mathfrak{m})| \\ (\mathfrak{q})_{k(\mathfrak{m})}^{w_{\mathfrak{m}}(N(\mathfrak{a})-1)/w_k} & \text{if } S(\mathfrak{m}) = \{\mathfrak{q}\}, \end{cases} \quad (2.1)$$

where  $N(\mathfrak{a}) := \#(\mathcal{O}_k/\mathfrak{a})$ , and where  $(\mathfrak{q})_{k(\mathfrak{m})}$  is the product of the prime ideals of  $\mathcal{O}_{k(\mathfrak{m})}$  which lie above  $\mathfrak{q}$ . Moreover, if  $\varphi_{\mathfrak{m}}(1)$  is the Robert-Ramachandra invariant, as defined in [9, p15], or in [3, p55], we have by [10, Corollaire 1.3, (iii)]

$$\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})^{12m} = \varphi_{\mathfrak{m}}(1)^{N(\mathfrak{a})-(\mathfrak{a}, k(\mathfrak{m})/k)}, \quad (2.2)$$

where  $m$  is the positive generator of  $\mathfrak{m} \cap \mathbb{Z}$ , and  $(\mathfrak{a}, k(\mathfrak{m})/k)$  is the Artin automorphism of  $k(\mathfrak{m})/k$  defined by  $\mathfrak{a}$ . If  $\mathfrak{a}$  is prime to  $6\mathfrak{m}\mathfrak{q}$ , then by [10, Corollaire 1.3, (ii-1)] we have

$$N_{k(\mathfrak{m}\mathfrak{q})/k(\mathfrak{m})}(\psi(1; \mathfrak{m}\mathfrak{q}, \mathfrak{a}^{-1}\mathfrak{m}\mathfrak{q}))^{w_{\mathfrak{m}}w_{\mathfrak{m}\mathfrak{q}}^{-1}} = \begin{cases} \psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})^{1-(\mathfrak{q}, k(\mathfrak{m})/k)^{-1}} & \text{if } \mathfrak{q} \nmid \mathfrak{m}, \\ \psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}) & \text{if } \mathfrak{q} \mid \mathfrak{m}. \end{cases} \quad (2.3)$$

**Definition 2.1** Let  $F \subseteq \mathbb{C}$  be an abelian extension of  $k$ , and write  $\mu(F)$  for the group of roots of unity in  $F$ . Let  $\mathfrak{m}$  be a nonzero proper ideal of  $\mathcal{O}_k$ . We define the  $\mathbb{Z}[\text{Gal}(F/k)]$ -submodule  $\Psi(F, \mathfrak{m})$  of  $F^\times$ , generated by the  $w_{\mathfrak{m}}$ -roots of all  $N_{k(\mathfrak{m})/k(\mathfrak{m}) \cap F}(\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}))$ , where  $\mathfrak{a}$  is any nonzero ideal of  $\mathcal{O}_k$  prime to  $6\mathfrak{m}$ . Also, we set  $\Psi'(F, \mathfrak{m}) := \mathcal{O}_F^\times \cap \Psi(F, \mathfrak{m})$ .

Then, we let  $C_F$  be the group generated by  $\mu(F)$  and by all  $\Psi'(F, \mathfrak{m})$ , for any nonzero proper ideal  $\mathfrak{m}$  of  $\mathcal{O}_k$ .

**Remark 2.1** If  $F' \subseteq \mathbb{C}$  is a finite extension of  $F$ , abelian over  $k$ , then for all nonzero proper ideal  $\mathfrak{m}$  of  $\mathcal{O}_k$ , we have  $N_{F'/F}(\Psi'(F', \mathfrak{m})) \subseteq \Psi'(F, \mathfrak{m})$  and  $\Psi'(F, \mathfrak{m}) \subseteq \Psi'(F', \mathfrak{m})$ . We deduce  $N_{F'/F}(C_{F'}) \subseteq C_F$  and  $C_F \subseteq C_{F'}$ .

From (2.1) we deduce immediately the following remark.

**Remark 2.2** If  $\mathfrak{m}$  is a nonzero ideal of  $\mathcal{O}_k$  which is divisible by at least two distinct primes, then  $\Psi'(F, \mathfrak{m}) = \Psi(F, \mathfrak{m})$ . If  $\mathfrak{q}$  is a maximal ideal and  $n \in \mathbb{N}^*$ , then  $\Psi'(F, \mathfrak{q}^n)$  is generated by the products  $\prod_{s \in S} N_{k(\mathfrak{q}^n)/k(\mathfrak{q}^n) \cap F}(\psi(1; \mathfrak{q}^n, \mathfrak{a}^{-1}\mathfrak{q}^n))^{\alpha_s \sigma_s}$ , where  $S$  is a finite set,  $(\mathfrak{a}_s)_{s \in S}$  is a family of nonzero ideals of  $\mathcal{O}_k$  which are prime to  $6\mathfrak{q}^n$ ,  $(\alpha_s)_{s \in S} \in \mathbb{Z}^S$  is such that  $\sum_{s \in S} \alpha_s (N(\mathfrak{a}_s) - 1) = 0$ , and  $(\sigma_s)_{s \in S} \in \text{Gal}(F/k)^S$ .

**Lemma 2.1** Let  $\mathfrak{m}$  and  $\mathfrak{g}$  be two nonzero proper ideals of  $\mathcal{O}_k$ , such that the conductor of  $F/k$  divides  $\mathfrak{m}$ . If  $\mathfrak{g} \wedge \mathfrak{m} = 1$ , then  $\Psi'(F, \mathfrak{g}) \subseteq C_F \cap \mathcal{O}_{k(1)}^\times$ . Else we have  $\Psi'(F, \mathfrak{g}) \subseteq \Psi'(F, \mathfrak{g} \wedge \mathfrak{m})$ .

*Proof.* If  $\mathfrak{g} \wedge \mathfrak{m} = 1$ , then  $k(\mathfrak{g}) \cap F \subseteq k(\mathfrak{g}) \cap k(\mathfrak{m}) = k(1)$ . Now assume  $\mathfrak{g}' := \mathfrak{g} \wedge \mathfrak{m} \neq 1$ . There are maximal ideals  $\mathfrak{q}_1, \dots, \mathfrak{q}_n$  of  $\mathcal{O}_k$  such that  $\mathfrak{g} = \mathfrak{g}' \mathfrak{q}_1 \cdots \mathfrak{q}_n$ . By recurrence, we are reduced to the case  $n = 1$ . Let  $\mathfrak{a}$  be a nonzero ideal which is prime to  $6\mathfrak{g}$ , and let  $x$  be a  $w_{\mathfrak{g}}$ -th root of  $N_{k(\mathfrak{g})/k(\mathfrak{g}) \cap F}(\psi(1; \mathfrak{g}, \mathfrak{a}^{-1}\mathfrak{g}))$ . As above,  $k(\mathfrak{g}) \cap F = k(\mathfrak{g}') \cap F$ , and then from (2.3) there is  $\alpha \in \mathbb{Z}[\text{Gal}(k(\mathfrak{m})/k)]$  such that

$$x^{w_{\mathfrak{g}'}} = N_{k(\mathfrak{g}')/k(\mathfrak{g}') \cap F} \left( N_{k(\mathfrak{g})/k(\mathfrak{g}')}(\psi(1; \mathfrak{g}, \mathfrak{a}^{-1}\mathfrak{g}))^{w_{\mathfrak{g}'}/w_{\mathfrak{g}}} \right) = N_{k(\mathfrak{g}')/k(\mathfrak{g}') \cap F}(\psi(1; \mathfrak{g}', \mathfrak{a}^{-1}\mathfrak{g}'))^\alpha.$$

□

**Corollary 2.1** *For any nonzero proper ideal  $\mathfrak{m}$  of  $\mathcal{O}_k$ , such that the conductor of  $F/k$  divides  $\mathfrak{m}$ , the group  $C_F$  is generated by  $\mu(F)$ , by  $C_F \cap \mathcal{O}_{k(1)}^\times$ , and by the  $\Psi'(F, \mathfrak{g})$ , where  $\mathfrak{g}$  is a nonzero proper ideal of  $\mathcal{O}_k$  which divides  $\mathfrak{m}$ .*

Let  $\mathfrak{f}$  be the ideal of  $\mathcal{O}_k$ , prime to  $\mathfrak{p}$ , such that the conductor of  $K_0/k$  divides  $\mathfrak{fp}^\infty$ . For  $n \in \mathbb{N}$ , and any ideal  $\mathfrak{g} \neq (0)$  of  $\mathcal{O}_k$ , we set

$$\Psi'(K_n, \mathfrak{gp}^\infty) := \bigcup_{m=1}^{\infty} \Psi'(K_n, \mathfrak{gp}^m).$$

Remark that in view of (2.3),  $\Psi'(K_n, \mathfrak{gp}^m) \subseteq \mu_{w_{\mathfrak{gp}^m}}(K_n) \Psi'(K_n, \mathfrak{gp}^{m+1})$  for any  $m \in \mathbb{N}$ , where for any  $j \in \mathbb{N}^*$  and any field  $L$ ,  $\mu_j(L)$  is the group of  $j$ -th roots of unity in  $L$ . From Corollary 2.1, we deduce the following remark.

**Remark 2.3** *The group  $C_n := C_{K_n}$  is generated by  $\mu(K_n)$ , by  $C_n \cap \mathcal{O}_{k(1)}^\times$ , by  $\Psi'(K_n, \mathfrak{p}^\infty)$ , and by the  $\Psi'(K_n, \mathfrak{g})$  and the  $\Psi'(K_n, \mathfrak{gp}^\infty)$ , where  $\mathfrak{g}$  is a nonzero proper ideal of  $\mathcal{O}_k$  which divides  $\mathfrak{f}$ .*

We define  $\mathcal{C}_n := \mathbb{Z}_p \otimes_{\mathbb{Z}} C_n$ ,  $\mathcal{C}_\infty := \varprojlim (\mathcal{C}_n)$ , and for any nonzero ideal  $\mathfrak{g}$  of  $\mathcal{O}_k$ , we set  $\overline{\Psi}'(K_\infty, \mathfrak{gp}^\infty) := \varprojlim (\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi'(K_n, \mathfrak{gp}^\infty))$ .

**Lemma 2.2** *The group  $\mathcal{C}_\infty$  is generated by the  $\overline{\Psi}'(K_\infty, \mathfrak{gp}^\infty)$ , with  $\mathfrak{g}$  a nonzero ideal of  $\mathcal{O}_k$  dividing  $\mathfrak{f}$ .*

*Proof.* Since  $\mu(K_\infty)$  is finite,  $\varprojlim (\mu(K_n))$  is trivial. For any proper ideal  $\mathfrak{g}$  dividing  $\mathfrak{f}$ ,  $\Psi'(K_n, \mathfrak{g}) \subseteq \mathcal{O}_{k(\mathfrak{g})}$  for all  $n \in \mathbb{N}$ , so one can easily check that  $\varprojlim (\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi'(K_n, \mathfrak{g}))$  is trivial. In the same way,  $\varprojlim (\mathbb{Z}_p \otimes_{\mathbb{Z}} (C_n \cap \mathcal{O}_{k(1)}^\times))$  is trivial. Then from Remark (2.3), we derive the lemma.  $\square$

**Definition 2.2** *We write  $C_F^{\mathbb{R}}$  for the subgroup of  $\mathcal{O}_F^\times$  generated by  $\mu(F)$  and by the elements  $N_{k(\mathfrak{m})/k(\mathfrak{m}) \cap F}(\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}))^{\sigma^{-1}}$ , where  $\mathfrak{m}$  is a nonzero proper ideal of  $\mathcal{O}_k$ ,  $\mathfrak{a}$  is a nonzero ideal of  $\mathcal{O}_k$ , prime to  $6\mathfrak{m}$ , and  $\sigma \in \text{Gal}(F/k)$ . This is the group used for instance in [13] and [7].*

**Remark 2.4** *It is well known that  $\mathcal{O}_F^\times / C_F^{\mathbb{R}}$  is finite. On the other hand, it is obvious that  $C_F^{\mathbb{R}} \subseteq C_F$ . Hence  $\mathcal{O}_F^\times / C_F$  and  $C_F / C_F^{\mathbb{R}}$  are finite.*

**Remark 2.5** *Using an additive notation, for  $n \in \mathbb{N}$ ,  $C_n^{\mathbb{R}} := C_{K_n}^{\mathbb{R}}$  is generated by  $\mu(K_n)$ ,  $C_n^{\mathbb{R}} \cap \mathcal{O}_{k(\mathfrak{f})}^\times$ ,  $\mathbb{Z}[G_n]_0 \Psi(K_n, \mathfrak{p}^\infty)$ ,  $\mathbb{Z}[G_n]_0 \Psi(K_n, \mathfrak{g})$  and  $\mathbb{Z}[G_n]_0 \Psi(K_n, \mathfrak{gp}^\infty)$ , where  $\mathfrak{g}$  is a nonzero proper ideal which divides  $\mathfrak{f}$ , and where  $\mathbb{Z}[G_n]_0$  is the augmentation ideal of  $\mathbb{Z}[G_n]$ .*

Let  $\mathbb{Q}^{\text{alg}}$  be an algebraic closure of  $\mathbb{Q}$ . We choose arbitrarily two embeddings  $\mathbb{Q}^{\text{alg}} \hookrightarrow \mathbb{C}$  and  $\mathbb{Q}^{\text{alg}} \hookrightarrow \mathbb{C}_p$ . For any finite abelian group  $\mathcal{G}$ , we denote by  $\widehat{\mathcal{G}}$  the set of irreducible  $\mathbb{Q}^{\text{alg}}$  characters on  $\mathcal{G}$ . For any  $\chi \in \widehat{\mathcal{G}}$ , we denote by  $e_\chi$  the idempotent attached to  $\chi$ , i.e

$$e_\chi = \#(\mathcal{G})^{-1} \sum_{g \in \mathcal{G}} \chi(g) g^{-1}.$$

**Lemma 2.3** *For  $n \in \mathbb{N}$ , we set  $C_n^{\mathbb{R}} := C_{K_n}^{\mathbb{R}}$  and  $\mathcal{C}_n^{\mathbb{R}} := \mathbb{Z}_p \otimes_{\mathbb{Z}} C_n^{\mathbb{R}}$ . For all  $n \in \mathbb{N}$ , the orders of  $\mathcal{C}_n / \mathcal{C}_n^{\mathbb{R}}$  are bounded.*

*Proof.* By Remark 2.3, we just have to show that  $\left| C_n \cap \mathcal{O}_{k(1)}^\times / C_n^\mathbb{R} \cap \mathcal{O}_{k(1)}^\times \right|$  is bounded, and that the

$$X_{n,\mathfrak{g}} := \mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi'(K_n, \mathfrak{g}) / (\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi'(K_n, \mathfrak{g})) \cap C_n^\mathbb{R},$$

have bounded orders independantly of  $n$  and of the nonzero proper ideals  $\mathfrak{g}$  of  $\mathcal{O}_k$  which divide  $\mathfrak{f}_p^\infty$ . The quotient  $C_n \cap \mathcal{O}_{k(1)}^\times / C_n^\mathbb{R} \cap \mathcal{O}_{k(1)}^\times$  has a bounded order because if  $n$  is large enough, it is a subgroup of a quotient of  $\mathcal{O}_{k(1) \cap K_\infty}^\times / C_{k(1) \cap K_\infty}^\mathbb{R}$ . In particular, there is  $r \in \mathbb{N}^*$  an integer such that for all  $n \in \mathbb{N}$ ,

$$\left| C_{k_n} \cap \mathcal{O}_{k(1)}^\times / C_{k_n}^\mathbb{R} \cap \mathcal{O}_{k(1)}^\times \right| \leq r. \quad (2.4)$$

For any nonzero proper ideal  $\mathfrak{g}$  of  $\mathcal{O}_k$  let  $g \in \mathbb{N}$  be such that  $\mathfrak{g} \cap \mathbb{Z} = g\mathbb{Z}$ . Since  $\mu_{p^\infty}(K_\infty) := \bigcup_{j=0}^\infty \mu_{p^j}(K_\infty)$  is finite, we are reduced to show that the

$$X'_{n,\mathfrak{g}} := 12gw_{\mathfrak{g}} \mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi'(K_n, \mathfrak{g}) / 12gw_{\mathfrak{g}} ((\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi'(K_n, \mathfrak{g})) \cap C_n^\mathbb{R})$$

have bounded orders. First, let us show that the minimal number of generators of the  $\mathbb{Z}_p$ -modules  $X'_{n,\mathfrak{g}}$  is bounded. We notice that  $X'_{n,\mathfrak{g}}$  is a quotient of

$$Y_{n,\mathfrak{g}} := 12gw_{\mathfrak{g}} \mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi'(K_n, \mathfrak{g}) / 12gw_{\mathfrak{g}} I_n (\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi(K_n, \mathfrak{g})) , \quad (2.5)$$

where  $I_n$  is the augmentation ideal of  $\mathbb{Z}_p[G_n]$ . From [15, Lemme 1.1, Chapitre IV] and (2.2), we deduce that

$$Y_{n,\mathfrak{g}} = \begin{cases} J_n \varphi_{n,\mathfrak{g}}(1) / I_n J_n \varphi_{n,\mathfrak{g}}(1) & \text{if } \mathfrak{g} \text{ is divisible by at least two distinct primes,} \\ (I_n \cap J_n) \varphi_{n,\mathfrak{g}}(1) / I_n J_n \varphi_{n,\mathfrak{g}}(1) & \text{if } \mathfrak{g} \text{ is a power of a maximal ideal,} \end{cases} \quad (2.6)$$

where  $\varphi_{n,\mathfrak{g}}(1) := N_{k(\mathfrak{g})/K_n \cap k(\mathfrak{g})}(\varphi_{\mathfrak{g}}(1))$ , and where  $J_n$  is the annihilator of the  $\mathbb{Z}_p[G_n]$ -module  $\mu_{p^\infty}(K_n)$ . By (2.6), it is enough to show that the minimal number of generators of the  $\mathbb{Z}_p$ -module  $J_n/I_n J_n$  is bounded. It suffices to prove that the kernel of the surjections  $J_n/I_n J_n \rightarrow J_0/I_0 J_0$ , is bounded. This kernel is included in  $I_n \cap J_n/I_n J_n$ , which is a quotient of  $I_\infty \cap J_\infty/I_\infty J_\infty$ , with  $I_\infty := \varprojlim (I_n)$  and  $J_\infty := \varprojlim (J_n)$ . But  $I_\infty \cap J_\infty/I_\infty J_\infty$  is a pseudo-nul  $\Lambda'$ -module, annihilated by  $\#(\mu_{p^\infty}(K_\infty))$  and  $\gamma - 1$ . Hence  $I_\infty \cap J_\infty/I_\infty J_\infty$  is finite. We deduce that the minimal number of generators of the  $\mathbb{Z}_p$ -modules  $X'_{n,\mathfrak{g}}$  is bounded.

Now in order to prove the lemma, we just have to find out some  $a \in \mathbb{Z}_p$  such that  $a$  annihilates all the  $X'_{n,\mathfrak{g}}$ . Let  $\zeta \in \mathbb{C}_p$  be a primitive  $[K_0 : k]$ -th root of unity, and let  $a \in \mathbb{Z}_p$  be such that  $a$  is divisible by

$$r[K_0 : k] \#(\mu_{p^\infty}(K_\infty)) \prod_{\substack{\chi \in \widehat{G}_{\mathfrak{f}} \\ \chi \neq 1}} (1 - \chi(\sigma_\chi))$$

in  $\mathbb{Z}_p[\zeta]$ , where for all nontrivial  $\chi \in \widehat{G}_{\mathfrak{f}}$ , we have arbitrarily chosen  $\sigma_\chi \in G_{\mathfrak{f}}$  such that  $\chi(\sigma_\chi) \neq 1$ . The canonical map  $X'_{n,\mathfrak{g}} \rightarrow \mathbb{Z}_p[\zeta] \otimes_{\mathbb{Z}_p} X'_{n,\mathfrak{g}}$  is injective, hence it is sufficient to show that  $a$  annihilates  $\overline{X}'_{n,\mathfrak{g}} := \mathbb{Z}_p[\zeta] \otimes_{\mathbb{Z}_p} X'_{n,\mathfrak{g}}$ . We have

$$a \overline{X}'_{n,\mathfrak{g}} \subseteq r \#(\mu_{p^\infty}(K_\infty)) [K_0 : k] e_1 \overline{X}'_{n,\mathfrak{g}} \oplus \bigoplus_{\substack{\chi \in \widehat{G}_{\mathfrak{f}} \\ \chi \neq 1}} (1 - \sigma_\chi) [K_0 : k] e_\chi \overline{X}'_{n,\mathfrak{g}}. \quad (2.7)$$



But obviously,  $(1 - \sigma_\chi)[K_0 : k]e_\chi \overline{X}'_{\mathfrak{g},n} = 0$  for all nontrivial  $\chi \in \widehat{G_f}$ . Then by (2.7) we just have to show that  $r\#(\mu_{p^\infty}(k_\infty))[K_0 : k]e_1 \overline{X}'_{n,\mathfrak{g}} = 0$ . By Remark 2.1, we have  $\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{k_n} \subseteq \mathcal{C}_n$  and

$$[K_0 : k]e_1 \mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi'(K_n, \mathfrak{g}) \subseteq \mathbb{Z}_p \otimes_{\mathbb{Z}} C_{k_n}. \quad (2.8)$$

As for (2.6), for all  $m \in \mathbb{N}^*$  we have

$$12p^m w_{\mathfrak{p}^m} \Psi'(k_n, \mathfrak{p}^m) / 12p^m w_{\mathfrak{p}^m} I'_n \Psi(k_n, \mathfrak{p}^m) = I'_n \cap J'_n \varphi'_{n,\mathfrak{p}^m}(1) / I'_n J'_n \varphi'_{n,\mathfrak{p}^m}(1), \quad (2.9)$$

where  $I'_n$  is the augmentation ideal of  $\mathbb{Z}[\Gamma/\Gamma_n]$ ,  $J'_n$  is the annihilator of the  $\mathbb{Z}[\Gamma/\Gamma_n]$ -module  $\mu_{p^\infty}(k_n)$ , and  $\varphi'_{n,\mathfrak{p}^m}(1) := N_{k(\mathfrak{p}^m)/k(\mathfrak{p}^m) \cap k_n}(\varphi_{\mathfrak{p}^m}(1))$ . By (2.9),  $\#(\mu_{p^\infty}(k_\infty))$  annihilates  $\Psi'(k_n, \mathfrak{p}^m) / I'_n \Psi(k_n, \mathfrak{p}^m)$ . Hence by (2.4) and by Corollary 2.1, we deduce that  $r\#(\mu_{p^\infty}(k_\infty))$  annihilates  $C_{k_n}/C_{k_n}^R$ . Then from the inclusion (2.8), we deduce the equality  $r\#(\mu_{p^\infty}(k_\infty))[K_0 : k]e_1 \overline{X}'_{n,\mathfrak{g}} = 0$ , which ends the proof of the lemma.  $\square$

**Corollary 2.2** *We set  $\mathcal{C}_\infty^R := \varprojlim (\mathcal{C}_n^R)$ . The canonical injection  $\mathcal{C}_\infty^R \hookrightarrow \mathcal{C}_\infty$  is a pseudo-isomorphism of  $\Lambda'$ -modules.*

*Proof.* From Lemma 2.3, there is  $a \in \mathbb{N}$  such that  $a$  annihilates  $\mathcal{C}_n/\mathcal{C}_n^R$  for all  $n \in \mathbb{N}$ . Then  $\mathcal{C}_\infty/\mathcal{C}_\infty^R$  is annihilated by  $a$  and by  $\gamma - 1$ , where  $\gamma$  is a topological generator of  $\Gamma$ , hence it is pseudo-nul.  $\square$

### 3 Elementary results.

Let  $F/k$  be an abelian extension of  $k$ . If  $[F : k] < \infty$ , we write  $\mathcal{K}_F$  for the pro- $p$ -completion of  $\prod_{\mathfrak{q}|\mathfrak{p}} F_{\mathfrak{q}}^\times$ , where for any prime ideal  $\mathfrak{q}$  of  $\mathcal{O}_F$  above  $\mathfrak{p}$ ,  $F_{\mathfrak{q}}$  is the completion of  $F$  at  $\mathfrak{q}$ .

When  $F/k$  is infinite, we define  $\mathcal{K}_F$  by taking projective limits over finite sub-extensions, under the norm maps. For all  $n \in \mathbb{N} \cup \{\infty\}$ , we write  $\mathcal{K}_n$  for  $\mathcal{K}_{K_n}$ . For any prime  $\mathfrak{q}$  of  $K_\infty$  over  $\mathfrak{p}$ , we define  $K_{\infty,\mathfrak{q}} := \bigcup_{n \in \mathbb{N}} K_{n,\mathfrak{q}}$ . One can verify that  $\mu_{p^\infty}(K_{\infty,\mathfrak{q}})$  is finite (otherwise see [17, Lemma 2.1]). Then by [6, Theorem 25], there is an exact sequence of  $\Lambda'$ -modules

$$0 \rightarrow \mathcal{K}_\infty \longrightarrow (\Lambda')^{[K_0:k]} \longrightarrow \bigoplus_{\mathfrak{q}|\mathfrak{p}} \mu_{p^\infty}(K_{\infty,\mathfrak{q}}) \rightarrow 0. \quad (3.1)$$

In particular  $\mathcal{K}_\infty$  is a finitely generated  $\Lambda'$ -module.

If  $[F : k] < \infty$ , we set  $\mathcal{E}_F := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F^\times$ , and we write  $\mathcal{U}_F$  for the pro- $p$ -completion of  $\prod_{\mathfrak{q}|\mathfrak{p}} \mathcal{O}_{F_{\mathfrak{q}}}^\times$ . The injection  $\mathcal{O}_F^\times \hookrightarrow \prod_{\mathfrak{q}|\mathfrak{p}} \mathcal{O}_{F_{\mathfrak{q}}}^\times$  induces a canonical map  $\mathcal{E}_F \rightarrow \mathcal{U}_F$ . The Leopoldt conjecture, which is known to be true for abelian extensions of  $k$ , states that this map is injective. When  $F/k$  is infinite, we define  $\mathcal{E}_F$ ,  $\mathcal{U}_F$  and a canonical injection  $\mathcal{E}_F \hookrightarrow \mathcal{U}_F$ , by taking projective limits over finite sub-extensions, under the norm maps. For all  $n \in \mathbb{N} \cup \{\infty\}$ , we set  $\mathcal{E}_n := \mathcal{E}_{K_n}$ , and  $\mathcal{U}_n := \mathcal{U}_{K_n}$ . Then  $\mathcal{E}_\infty$  and  $\mathcal{U}_\infty$  are two submodules of  $\mathcal{K}_\infty$ , hence are finitely generated over  $\Lambda'$ .

We set  $B_F := \text{Gal}(M_{\mathfrak{p}}(F)/F)$ , where we write  $M_{\mathfrak{p}}(F)$  for the maximal abelian pro- $p$ -extension of  $F$  which is unramified outside the primes above  $\mathfrak{p}$ . If  $[F : k] < \infty$ , let  $A_F$  be the  $p$ -part of the class group  $\text{Cl}(\mathcal{O}_F)$ . Else, we set  $A_F := \varprojlim A_{F'}$ , where the projective limit is taken over the finite sub-extensions of  $F/k$ , with respect to the norm maps. By class field theory,  $A_F$  is identified to the Galois group of the maximal abelian unramified

pro- $p$ -extension of  $F$ . For all  $n \in \mathbb{N} \cup \{\infty\}$ , we set  $A_n := A_{K_n}$ , and  $B_n := B_{K_n}$ . From [6, end of §3.2], we know that  $B_\infty$  is a finitely generated  $\Lambda'$ -module, hence it is also the case for  $A_\infty$ . From class field theory we have a map  $\mathcal{U}_F \rightarrow B_F$ , and then the following sequence of  $\mathbb{Z}_p[\text{Gal}(F/k)]$ -modules, which is called the inertia sequence, is exact,

$$0 \longrightarrow \mathcal{E}_F \longrightarrow \mathcal{U}_F \longrightarrow B_F \longrightarrow A_F \longrightarrow 0. \quad (3.2)$$

If  $[F : k] < \infty$ , we set  $\mathcal{E}'_F := \mathbb{Z}_p \otimes_{\mathbb{Z}} E'_F$ , where  $E'_F$  is the group of elements of  $F$  which are units outside of the primes above  $\mathfrak{p}$ . When  $F/k$  is infinite, we define  $\mathcal{E}'_F$  by taking projective limits over finite sub-extensions, under the norm maps. We denote by  $A'_F$  the Galois group of the maximal abelian unramified pro- $p$ -extension of  $F$  which is totally split above  $\mathfrak{p}$ . For all  $n \in \mathbb{N} \cup \{\infty\}$ , we write  $\mathcal{E}'_n$  for  $\mathcal{E}'_{K_n}$ , and we write  $A'_n$  for  $A'_{K_n}$ . The module  $A'_\infty$  is a quotient of  $B_\infty$ , hence a finitely generated  $\Lambda'$ -module. As above, by the Leopoldt conjecture we have a natural injection  $\mathcal{E}'_F \rightarrow \mathcal{K}_F$ . Then  $\mathcal{E}'_\infty$  is finitely generated over  $\Lambda'$ , since it's a submodule of  $\mathcal{K}_\infty$ . From class field theory we have a map  $\mathcal{K}_F \rightarrow B_F$ . Then the following sequence of  $\mathbb{Z}_p[[\text{Gal}(F/k)]]$ -modules, which is called the decomposition sequence, is exact,

$$0 \longrightarrow \mathcal{E}'_F \longrightarrow \mathcal{K}_F \longrightarrow B_F \longrightarrow A'_F \longrightarrow 0. \quad (3.3)$$

For any  $\Lambda'$ -module  $M$ , we denote by  $M^{\Gamma_n}$  the module of  $\Gamma_n$ -invariants of  $M$ , and we denote by  $M_{\Gamma_n}$  the module of  $\Gamma_n$ -coinvariants of  $M$ . By definition, they are respectively the kernel and the cokernel of the multiplication by  $1 - \gamma_n$  on  $M$ , where  $\gamma_n := \gamma^{p^n}$ . We recall that if  $M$  is finitely generated, then  $M^{\Gamma_n}$  is finite if and only if  $M_{\Gamma_n}$  is finite.

As in [1, Proposition 2.2], we have for all  $n \in \mathbb{N}$ , the exact sequence of  $\Lambda$ -modules

$$0 \longrightarrow (\mathcal{K}_\infty)_{\Gamma_n} \longrightarrow \mathcal{K}_n \longrightarrow \bigoplus_{\mathfrak{q}|\mathfrak{p}} \text{Gal}(K_{\infty, \mathfrak{q}'}/K_{n, \mathfrak{q}}) \rightarrow 0, \quad (3.4)$$

where for any prime  $\mathfrak{q}$  of  $K_\infty$  over  $\mathfrak{p}$ ,  $\mathfrak{q}'$  is an arbitrary prime of  $K_\infty$  above  $\mathfrak{q}$ .

Let  $M_\infty := \varprojlim (M_n)$ , with  $M_n$  a  $\mathbb{Z}_p[\Gamma/\Gamma_n]$ -module for every  $n \in \mathbb{N}$ . For such a  $\Lambda'$ -module, let us denote by  $\text{Ker}_n(M_\infty)$  and  $\text{Cok}_n(M_\infty)$  the kernel and cokernel of the natural map  $(M_\infty)_{\Gamma_n} \rightarrow M_n$ , and let us denote by  $\widetilde{M}_n$  the image of  $M_\infty$  in  $M_n$ .

**Lemma 3.1** *For all  $n \in \mathbb{N}$ ,  $\text{Ker}_n(\mathcal{E}'_\infty) = 0$ .*

*Proof.* Let  $n \in \mathbb{N}$ . The  $\mathbb{Z}_p$ -rank of  $\mathcal{E}_n$ ,  $\mathcal{U}_n$ , and  $A_n$  are respectively  $[K_n : k] - 1$ ,  $[K_n : k]$ , and 0. From the inertia sequence (3.2) applied to  $K_n$ , we deduce that the  $\mathbb{Z}_p$ -rank of  $B_n$  is 1. We have  $(B_\infty)_{\Gamma_n} = \text{Gal}(M_{\mathfrak{p}}(K_n)/K_\infty)$ , hence from the exact sequence below

$$0 \longrightarrow \text{Gal}(M_{\mathfrak{p}}(K_n)/K_\infty) \longrightarrow B_n \longrightarrow \Gamma_n \longrightarrow 0,$$

we deduce that the  $\mathbb{Z}_p$ -rank of  $(B_\infty)_{\Gamma_n}$  is 0. Then  $(B_\infty)_{\Gamma_n}$  and  $(B_\infty)^{\Gamma_n}$  are finite. By [4, end of section 4], we know that  $B_\infty$  has no nontrivial finite  $\Lambda'$ -submodule, and we deduce

$$(B_\infty)^{\Gamma_n} = 0. \quad (3.5)$$

By (3.3), if we denote by  $\mathcal{K}'_\infty$  the image of  $\mathcal{K}_\infty$  in  $B_\infty$ , we have the exact sequence

$$0 \rightarrow (\mathcal{E}'_\infty)^{\Gamma_n} \rightarrow (\mathcal{K}_\infty)^{\Gamma_n} \rightarrow (\mathcal{K}'_\infty)^{\Gamma_n} \rightarrow (\mathcal{E}'_\infty)_{\Gamma_n} \rightarrow (\mathcal{K}_\infty)_{\Gamma_n} \rightarrow (\mathcal{K}'_\infty)_{\Gamma_n} \rightarrow 0. \quad (3.6)$$



From (3.5) we have  $(\mathcal{K}'_\infty)^{\Gamma_n} = 0$ . The lemma follows from the diagram below, where the maps are injective by (3.6), (3.4), (3.3),

$$\begin{array}{ccc} (\mathcal{E}'_\infty)_{\Gamma_n} & \xrightarrow{\quad} & \mathcal{E}'_n \\ \downarrow & & \downarrow \\ (\mathcal{K}_\infty)_{\Gamma_n} & \hookrightarrow & \mathcal{K}_n \end{array}$$

□

Let  $\mathcal{D}_{\mathfrak{p}}$  be the decomposition group of  $\mathfrak{p}$  in  $K_\infty/k$ . For all  $n \in \mathbb{N}$ , we denote by  $D_n$  the subfield of  $K_\infty$  fixed by  $\mathcal{D}_{\mathfrak{p}}\Gamma_n$ . We set  $D_\infty := \bigcup_{n \in \mathbb{N}} D_n$  (remark that for  $n$  large enough,  $D_n = D_\infty$ ). For any  $\mathbb{Z}_p$ -module  $M$ , we denote by  $M_{\text{tor}}$  the submodule of  $M$  defined by the elements  $x \in M$  such that  $p^n x = 0$  for large enough  $n \in \mathbb{N}$ . Let us denote by  $N'_{K_n/D_n}$  the composite of  $N_{K_n/D_n} : \mathcal{U}_n \rightarrow \mathcal{U}_{D_n}$  with  $\mathcal{U}_{D_n} \rightarrow \mathcal{U}_{D_n}/(\mathcal{U}_{D_n})_{\text{tor}}$  (remark that  $(\mathcal{U}_{D_n})_{\text{tor}}$  is trivial if  $p \neq 2$ ).

For any abelian extension of local fields  $L'/L$ , and all  $x \in L'^\times$ , we write  $(x, L'/L)$  for the (local) norm residue symbol. We denote by  $\mathcal{O}_L^{\times,1}$  the group of units of  $\mathcal{O}_L$  which are congruent to 1 modulo the maximal ideal.

**Lemma 3.2** *Let  $n \in \mathbb{N}$  be such that  $D_\infty \subseteq K_n$ , let  $\mathfrak{q}$  be a prime of  $K_\infty$  lying above  $\mathfrak{p}$ , and let  $u \in \mathcal{O}_{K_{n,\mathfrak{q}}}^{\times,1}$ . Let us identify  $\mathcal{U}_n$  to  $\prod_{\mathfrak{r}|\mathfrak{p}} \mathcal{O}_{K_{n,\mathfrak{r}}}^{\times,1}$ , and let us write  $\tilde{\mathcal{U}}_n(\mathfrak{q})$  for the image of  $\tilde{\mathcal{U}}_n$  in  $\mathcal{O}_{K_{n,\mathfrak{q}}}^{\times,1}$ . Then  $u \in \tilde{\mathcal{U}}_n(\mathfrak{q})$  if and only if  $N_{K_{n,\mathfrak{q}}/k_{\mathfrak{q}}}(u) \in \bigcap_{m=0}^{\infty} N_{k_{m,\mathfrak{q}}/k_{\mathfrak{q}}}(\mathcal{O}_{k_{m,\mathfrak{q}}}^{\times,1})$ .*

*Proof.* By compacity of the  $\mathcal{U}_m$ , we have

$$\tilde{\mathcal{U}}_n = \bigcap_{n \leq m} N_{K_m/K_n}(\mathcal{U}_m). \quad (3.7)$$

If  $u \in \tilde{\mathcal{U}}_n(\mathfrak{q})$  then  $N_{K_{n,\mathfrak{q}}/k_{\mathfrak{q}}}(u) \in \bigcap_{m=0}^{\infty} N_{k_{m,\mathfrak{q}}/k_{\mathfrak{q}}}(\mathcal{O}_{k_{m,\mathfrak{q}}}^{\times,1})$ . Reciprocally, assume that we have  $N_{K_{n,\mathfrak{q}}/k_{\mathfrak{q}}}(u) \in \bigcap_{m=0}^{\infty} N_{k_{m,\mathfrak{q}}/k_{\mathfrak{q}}}(\mathcal{O}_{k_{m,\mathfrak{q}}}^{\times,1})$ . Then for  $m \geq n$ , we have  $(N_{K_{n,\mathfrak{q}}/k_{\mathfrak{q}}}(u), k_{m,\mathfrak{q}}/k_{\mathfrak{q}}) = 1$ , which implies  $(u, K_{m,\mathfrak{q}}/K_{n,\mathfrak{q}})_{|k_{m,\mathfrak{q}}} = 1$ . But  $\text{Gal}(K_m/K_n) \simeq \text{Gal}(k_m/k_n)$ , so the restriction map  $\text{Gal}(K_{m,\mathfrak{q}}/K_{n,\mathfrak{q}}) \rightarrow \text{Gal}(k_{m,\mathfrak{q}}/k_{n,\mathfrak{q}})$  is injective. Then we deduce  $(u, K_{m,\mathfrak{q}}/K_{n,\mathfrak{q}}) = 1$ , and  $u \in N_{K_{m,\mathfrak{q}}/K_{n,\mathfrak{q}}}(\mathcal{O}_{K_{m,\mathfrak{q}}}^{\times,1})$ . Now the lemma follows from (3.7). □

**Lemma 3.3** *Let  $\mathfrak{q}$  be a prime of  $K_\infty$  lying above  $\mathfrak{p}$ . Then*

$$\bigcap_{m=0}^{\infty} N_{k_{m,\mathfrak{q}}/k_{\mathfrak{q}}}(\mathcal{O}_{k_{m,\mathfrak{q}}}^{\times,1}) = \begin{cases} \{1\} & \text{if } p \neq 2, \\ \mu_2 & \text{if } p = 2. \end{cases}$$

*Proof.* We set  $\mathcal{N} := \bigcap_{m=0}^{\infty} N_{k_{m,\mathfrak{q}}/k_{\mathfrak{q}}}(\mathcal{O}_{k_{m,\mathfrak{q}}}^{\times,1})$ . By local class field theory, the inertia group of  $k_{\infty,\mathfrak{q}}/k_{\mathfrak{p}}$  is isomorphic to  $\mathcal{O}_{k_{\mathfrak{p}}}^{\times,1}/\mathcal{N}$ . Since  $k_{\infty,\mathfrak{q}}/k_{\mathfrak{p}}$  is infinitely ramified, we deduce that  $\mathcal{N}$  is a finite subgroup of  $\mathcal{O}_{k_{\mathfrak{p}}}^{\times,1}$ . Then  $\mathcal{N} = \{1\}$  if  $p \neq 2$ , and  $\mathcal{N} \subseteq \mu_2$  if  $p = 2$ . Assume  $p = 2$ . We have  $(-1, k_{\infty,\mathfrak{q}}/k_{\mathfrak{p}}) = 1$  since  $\Gamma$  is torsion-free. By class field theory, we deduce  $-1 \in \mathcal{N}$ . □

For all  $n \in \mathbb{N}$  we define a valuation map

$$\nu_n : \bigoplus_{\mathfrak{q}|\mathfrak{p} \text{ in } K_n} K_{n,\mathfrak{q}}^\times \longrightarrow \bigoplus_{\mathfrak{q}|\mathfrak{p} \text{ in } K_n} \mathbb{Z}\mathfrak{q}, \quad x = (x_{\mathfrak{q}})_{\mathfrak{q}|\mathfrak{p}} \mapsto \sum_{\mathfrak{q}|\mathfrak{p}} \nu_{\mathfrak{q}}(x_{\mathfrak{q}}) \mathfrak{q},$$

where  $\nu_{\mathfrak{q}}$  is the normalized valuation of  $K_{n,\mathfrak{q}}$ . We define  $\bar{\nu}_n : \mathcal{K}_n \rightarrow \bigoplus_{\mathfrak{q}|\mathfrak{p} \text{ in } K_n} \mathbb{Z}_p\mathfrak{q}$  by pro- $p$ -completion, and we define  $\bar{\nu}_\infty : \mathcal{K}_\infty \rightarrow \bigoplus_{\mathfrak{q}|\mathfrak{p} \text{ in } K_\infty} \mathbb{Z}_p\mathfrak{q}$  by taking projective limits. Then we have the following exact sequence of  $\Lambda$ -modules, for all  $n \in \mathbb{N} \cup \{\infty\}$ .

$$0 \longrightarrow \mathcal{U}_n \longrightarrow \mathcal{K}_n \xrightarrow{\bar{\nu}_n} \bigoplus_{\mathfrak{q}|\mathfrak{p} \text{ in } K_n} \mathbb{Z}_p\mathfrak{q} \longrightarrow 0 \quad (3.8)$$

**Proposition 3.1** *For all  $n \in \mathbb{N}$ , the  $\mathbb{Z}_p$ -rank of  $\text{Ker}_n(\mathcal{U}_\infty)$  and  $\text{Cok}_n(\mathcal{U}_\infty)$  is  $[D_n : k]$ , and we have an exact sequence of  $\Lambda$ -modules*

$$0 \longrightarrow \tilde{\mathcal{U}}_n \longrightarrow \mathcal{U}_n \xrightarrow{N'_{K_n/D_n}} \mathcal{U}_{D_n}/(\mathcal{U}_{D_n})_{\text{tor}}. \quad (3.9)$$

*Proof.* From (3.1) we know that  $(\mathcal{K}_\infty)^{\Gamma_n} = 0$ . Then from (3.8) we deduce the exact sequence below,

$$0 \longrightarrow \left( \bigoplus_{\mathfrak{q}|\mathfrak{p} \text{ in } K_\infty} \mathbb{Z}_p\mathfrak{q} \right)^{\Gamma_n} \longrightarrow (\mathcal{U}_\infty)_{\Gamma_n} \longrightarrow (\mathcal{K}_\infty)_{\Gamma_n} \longrightarrow \left( \bigoplus_{\mathfrak{q}|\mathfrak{p} \text{ in } K_\infty} \mathbb{Z}_p\mathfrak{q} \right)_{\Gamma_n} \longrightarrow 0. \quad (3.10)$$

From (3.10) and (3.4) we deduce that the  $\mathbb{Z}_p$ -rank of  $\text{Ker}_n(\mathcal{U}_\infty)$  is  $[D_n : k]$ .

Let us show that the sequence (3.9) is exact. We choose  $m \geq n$  such that  $D_\infty = D_m$ . As in [1, Proof of Proposition 2.3], the norm map  $\mathcal{U}_m \rightarrow \mathcal{U}_n$  is surjective. Then,  $D_m/D_n$  being totally split at the primes above  $\mathfrak{p}$ , we have

$$\begin{aligned} (\mathcal{U}_{D_n})_{\text{tor}} \cap N_{K_n/D_n}(\mathcal{U}_n) &= (\mathcal{U}_{D_n})_{\text{tor}} \cap N_{K_m/D_n}(\mathcal{U}_m) \\ &= N_{D_m/D_n}((\mathcal{U}_{D_m})_{\text{tor}} \cap N_{K_m/D_m}(\mathcal{U}_m)). \end{aligned} \quad (3.11)$$

By Lemma 3.2 and Lemma 3.3, we have

$$(\mathcal{U}_{D_m})_{\text{tor}} \cap N_{K_m/D_m}(\mathcal{U}_m) = N_{K_m/D_m}(\tilde{\mathcal{U}}_m). \quad (3.12)$$

From (3.11) and (3.12), we deduce

$$(\mathcal{U}_{D_n})_{\text{tor}} \cap N_{K_n/D_n}(\mathcal{U}_n) = N_{K_m/D_n}(\tilde{\mathcal{U}}_m) = N_{K_n/D_n}(\tilde{\mathcal{U}}_n),$$

which proves the exactness of the sequence (3.9). Since  $\mathcal{U}_{D_n}/N_{K_n/D_n}(\mathcal{U}_n)$  is a torsion  $\mathbb{Z}_p$ -module, we deduce that the  $\mathbb{Z}_p$ -rank of  $\text{Cok}_n(\mathcal{U}_\infty)$  is  $[D_n : k]$ .  $\square$

## 4 Descent kernels and cokernels.

For all abelian extension  $F/k$ , we set  $\mathcal{U}_F^{(0)} := N_{F/k}^{-1}((\mathcal{U}_k)_{\text{tor}})$  (so  $\mathcal{U}_F^{(0)} = \text{Ker}(N_{F/k})$  if  $p \neq 2$ ). Then one can easily check that

$$\mathcal{U}_F^{(0)}/\mathcal{E}_F = (\mathcal{U}_F/\mathcal{E}_F)_{\text{tor}} \quad \text{and} \quad \mathcal{U}_F^{(0)}/\mathcal{C}_F = (\mathcal{U}_F/\mathcal{C}_F)_{\text{tor}}. \quad (4.1)$$

For all  $n \in \mathbb{N}$ , we set  $\mathcal{U}_n^{(0)} := \mathcal{U}_{K_n}^{(0)}$ , and  $\mathcal{U}_\infty^{(0)} := \varprojlim \mathcal{U}_n^{(0)}$ . By Proposition 3.1, we have  $\tilde{\mathcal{U}}_n \subseteq \mathcal{U}_n^{(0)}$  and therefore  $\mathcal{U}_\infty^{(0)} = \mathcal{U}_\infty$ . However take care for example that for  $n \in \mathbb{N}$ ,  $\text{Ker}_n(\mathcal{U}_\infty^{(0)}/\mathcal{E}_\infty)$  may not be equal to  $\text{Ker}_n(\mathcal{U}_\infty/\mathcal{E}_\infty)$ .

**Lemma 4.1** *The orders of  $\text{Ker}_n(\mathcal{U}_\infty^{(0)}/\mathcal{E}_\infty)$  and of  $\text{Cok}_n(\mathcal{U}_\infty^{(0)}/\mathcal{E}_\infty)$  are finite and bounded.*

*Proof.* By (3.5) the inertia sequence (3.2) gives the exact sequence below,

$$0 \longrightarrow (A_\infty)^{\Gamma_n} \longrightarrow (\mathcal{U}_\infty/\mathcal{E}_\infty)_{\Gamma_n} \longrightarrow (B_\infty)_{\Gamma_n} \longrightarrow (A_\infty)_{\Gamma_n} \longrightarrow 0. \quad (4.2)$$

Moreover, since  $\mathcal{U}_n^{(0)}/\mathcal{E}_n \simeq (\mathcal{U}_n/\mathcal{E}_n)_{\text{tor}}$  we deduce from (3.2) the exact sequence below,

$$0 \longrightarrow \mathcal{U}_n^{(0)}/\mathcal{E}_n \longrightarrow (B_n)_{\text{tor}} \longrightarrow A_n. \quad (4.3)$$

We have  $(B_\infty)_{\Gamma_n} \simeq \text{Gal}(\mathbb{M}_{\mathfrak{p}}(K_n)/K_\infty) = (B_n)_{\text{tor}}$ . Then we apply the snake lemma to the diagram below

$$\begin{array}{ccccccc} 0 & \longrightarrow & (B_\infty)_{\Gamma_n} & \longrightarrow & (B_n)_{\text{tor}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Ker}_n(A_\infty) & \longrightarrow & (A_\infty)_{\Gamma_n} & \longrightarrow & A_n \end{array}, \quad (4.4)$$

and by (4.2) and (4.3) we obtain the following exact sequence

$$0 \longrightarrow (A_\infty)^{\Gamma_n} \longrightarrow (\mathcal{U}_\infty/\mathcal{E}_\infty)_{\Gamma_n} \longrightarrow \mathcal{U}_n^{(0)}/\mathcal{E}_n \longrightarrow \text{Ker}_n(A_\infty) \longrightarrow 0. \quad (4.5)$$

As in [12, Proof of theorem 1.4], one can prove that  $(A_\infty)_{\Gamma_n}$  and  $(A_\infty)^{\Gamma_n}$  are finite, and then from (4.5) we deduce that the orders of  $\text{Ker}_n(\mathcal{U}_\infty^{(0)}/\mathcal{E}_\infty)$  and of  $\text{Cok}_n(\mathcal{U}_\infty^{(0)}/\mathcal{E}_\infty)$  are finite. Moreover  $A_\infty$  is noetherian over  $\Lambda'$ , so the sequence  $(\text{Ker}_n(\mathcal{U}_\infty^{(0)}/\mathcal{E}_\infty))_{n \in \mathbb{N}}$  stabilizes. For  $n$  large enough (such that  $K_\infty/K_n$  is totally ramified at the primes above  $\mathfrak{p}$ ), and for all  $m \in \mathbb{N}$ , one can easily check that  $A_{n+m} \rightarrow A_n$  is surjective, and then

$$\text{Cok}_n(A_\infty) = 0. \quad (4.6)$$

Moreover by a classical theorem of Iwasawa, there exists  $c_1 \in \mathbb{Z}$  such that for  $n$  large enough,  $\#(A_n) = p^{\mu(A_\infty)p^n + \lambda(A_\infty)n + c_1}$ , where  $\lambda(A_\infty)$  and  $\mu(A_\infty)$  are respectively the  $\lambda$ -invariant and the  $\mu$ -invariant of  $A_\infty$ . Since  $(A_\infty)_{\Gamma_m}$  is finite for all  $m \in \mathbb{N}$ , it is well known, from the general theory of finitely generated  $\Lambda'$ -modules, that there exists  $c_2 \in \mathbb{Z}$  such that  $\#((A_\infty)_{\Gamma_n}) = p^{\mu(A_\infty)p^n + \lambda(A_\infty)n + c_2}$  for  $n$  large enough. From (4.6), we deduce that for  $n$  large enough,  $\#(\text{Ker}_n(A_\infty)) = p^{c_2 - c_1}$ , i.e  $\#(\text{Cok}_n(\mathcal{U}_\infty^{(0)}/\mathcal{E}_\infty)) = p^{c_2 - c_1}$  by (4.5).  $\square$

**Lemma 4.2** *The modules  $\mathcal{U}_\infty/\mathcal{C}_\infty$ ,  $\mathcal{U}_\infty/\mathcal{E}_\infty$  and  $\mathcal{E}_\infty/\mathcal{C}_\infty$  are torsion over  $\Lambda'$ , and their characteristic ideals are prime to  $1 - \gamma_n$ , for all  $n \in \mathbb{N}$ .*

*Proof.* It is enough to show the lemma for  $\mathcal{U}_\infty/\mathcal{C}_\infty$ , since  $\mathcal{U}_\infty/\mathcal{E}_\infty$  is a quotient of  $\mathcal{U}_\infty/\mathcal{C}_\infty$ , and since  $\mathcal{E}_\infty/\mathcal{C}_\infty$  is a submodule of  $\mathcal{U}_\infty/\mathcal{C}_\infty$ . We refer the reader to [17, Proposition 3.1] and [16, Theorem 6.1].  $\square$

**Lemma 4.3** *The orders of  $\text{Ker}_n(\mathcal{U}_\infty^{(0)}/\mathcal{C}_\infty)$  and of  $\text{Ker}_n(\mathcal{E}_\infty/\mathcal{C}_\infty)$  are finite and asymptotically equivalent. The orders of  $\text{Cok}_n(\mathcal{U}_\infty^{(0)}/\mathcal{C}_\infty)$  and of  $\text{Cok}_n(\mathcal{E}_\infty/\mathcal{C}_\infty)$  are finite and asymptotically equivalent.*

*Proof.* From (3.2) we deduce that  $(\mathcal{U}_\infty^{(0)}/\mathcal{E}_\infty)^{\Gamma_n}$  is isomorphic to a submodule of  $(B_\infty)^{\Gamma_n}$ , hence is zero by (3.5). Then we have the following sequence,

$$0 \longrightarrow (\mathcal{E}_\infty/\mathcal{C}_\infty)_{\Gamma_n} \longrightarrow (\mathcal{U}_\infty^{(0)}/\mathcal{C}_\infty)_{\Gamma_n} \longrightarrow (\mathcal{U}_\infty^{(0)}/\mathcal{E}_\infty)_{\Gamma_n} \longrightarrow 0,$$

and we deduce the exact sequence below,

$$\begin{aligned} 0 \longrightarrow \text{Ker}_n(\mathcal{E}_\infty/\mathcal{C}_\infty) &\longrightarrow \text{Ker}_n(\mathcal{U}_\infty^{(0)}/\mathcal{C}_\infty) \longrightarrow \text{Ker}_n(\mathcal{U}_\infty^{(0)}/\mathcal{E}_\infty) \longrightarrow \cdots \\ \cdots &\longrightarrow \text{Cok}_n(\mathcal{E}_\infty/\mathcal{C}_\infty) \longrightarrow \text{Cok}_n(\mathcal{U}_\infty^{(0)}/\mathcal{C}_\infty) \longrightarrow \text{Cok}_n(\mathcal{U}_\infty^{(0)}/\mathcal{E}_\infty) \longrightarrow 0. \end{aligned} \quad (4.7)$$

By Lemma 4.2,  $(\mathcal{U}_\infty^{(0)}/\mathcal{C}_\infty)_{\Gamma_n}$  and  $(\mathcal{E}_\infty/\mathcal{C}_\infty)_{\Gamma_n}$  are finite. From Remark 2.4, the group  $\mathcal{E}_n/\mathcal{C}_n$  is also finite. Then Lemma 4.3 follows from (4.7) and Lemma 4.1.  $\square$

**Proposition 4.1** *The orders of  $\text{Ker}_n(\mathcal{U}_\infty^{(0)}/\mathcal{C}_\infty)$  and of  $\text{Ker}_n(\mathcal{E}_\infty/\mathcal{C}_\infty)$  are bounded.*

*Proof.* By Lemma 4.3, we just have to show that the orders of  $\text{Ker}_n(\mathcal{E}_\infty/\mathcal{C}_\infty)$  are bounded. For any  $\Lambda'$ -module  $M$ , we set  $I(M) := \bigcup_{n \in \mathbb{N}} M^{\Gamma_n}$ . Since the natural map  $\mathcal{E}'_\infty \rightarrow \mathcal{K}_\infty$  is injective by the Leopoldt conjecture, we have  $(\mathcal{E}'_\infty)^{\Gamma_n} = 0$  by (3.1). We deduce the following exact sequence,

$$0 \longrightarrow (\mathcal{E}'_\infty/\mathcal{C}_\infty)^{\Gamma_n} \longrightarrow (\mathcal{C}_\infty)_{\Gamma_n} \longrightarrow (\mathcal{E}'_\infty)_{\Gamma_n} \longrightarrow (\mathcal{E}'_\infty/\mathcal{C}_\infty)_{\Gamma_n} \longrightarrow 0. \quad (4.8)$$

By Lemma 3.1 and (4.8), we deduce  $\text{Ker}_n(\mathcal{C}_\infty) \simeq (\mathcal{E}'_\infty/\mathcal{C}_\infty)^{\Gamma_n}$ . Since  $\mathcal{E}'_\infty/\mathcal{C}_\infty$  is noetherian over  $\Lambda'$ , we deduce that for  $n \in \mathbb{N}$  large enough, we have

$$\text{Ker}_n(\mathcal{C}_\infty) \simeq I(\mathcal{E}'_\infty/\mathcal{C}_\infty).$$

In the same way, for  $n$  large enough we have  $\text{Ker}_n(\mathcal{E}_\infty) \simeq I(\mathcal{E}'_\infty/\mathcal{E}_\infty)$ , and then

$$\text{Ker}_n(\mathcal{E}_\infty)/\text{Im}(\text{Ker}_n(\mathcal{C}_\infty)) \simeq I(\mathcal{E}'_\infty/\mathcal{E}_\infty)/\text{Im}(I(\mathcal{E}'_\infty/\mathcal{C}_\infty)). \quad (4.9)$$

From Lemma 2.2 and (2.3), we deduce that  $\tilde{\mathcal{C}}_n$  is the product of the  $\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi'(K_n, \mathfrak{g}^{\infty})$ , where  $\mathfrak{g} \neq (0)$  is an ideal of  $\mathcal{O}_k$  dividing  $\mathfrak{f}$ . From Remark 2.3, for all  $n \in \mathbb{N}$ ,  $\mathcal{C}_n \rightarrow \text{Cok}_n(\mathcal{C}_\infty)$  restricts into a surjection from  $\mathcal{S}_n := \mathcal{C}_n \cap \mathcal{E}_{k(\mathfrak{f})}$  onto  $\text{Cok}_n(\mathcal{C}_\infty)$ . Since  $\mathcal{E}_{k(\mathfrak{f})}$  is noetherian, there is  $M \in \mathbb{N}$  such that for  $n$  large enough,  $\mathcal{S}_n = \mathcal{S}_M$ . Now  $(\mathcal{S}_M \cap \tilde{\mathcal{C}}_n)_{n \in \mathbb{N}}$  is an increasing sequence of submodules of  $\mathcal{S}_M$ , which is noetherian. Hence there exists a submodule  $\mathcal{S}'$  of  $\mathcal{S}_M$  such that, for  $n$  large enough,  $\mathcal{C}_M \hookrightarrow \mathcal{C}_n$  gives an isomorphism

$$\mathcal{S}_M/\mathcal{S}' \simeq \text{Cok}_n(\mathcal{C}_\infty). \quad (4.10)$$

Applying the snake lemma to the diagram below,

$$\begin{array}{ccccccc} & & (\mathcal{C}_\infty)_{\Gamma_n} & \longrightarrow & (\mathcal{E}_\infty)_{\Gamma_n} & \longrightarrow & (\mathcal{E}_\infty/\mathcal{C}_\infty)_{\Gamma_n} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{C}_n & \longrightarrow & \mathcal{E}_n & \longrightarrow & \mathcal{E}_n/\mathcal{C}_n \longrightarrow 0, \end{array}$$

and since  $\text{Ker}_n(\mathcal{E}_\infty/\mathcal{C}_\infty)$  is finite, we obtain the following exact sequence,

$$\text{Ker}_n(\mathcal{C}_\infty) \longrightarrow \text{Ker}_n(\mathcal{E}_\infty) \longrightarrow \text{Ker}_n(\mathcal{E}_\infty/\mathcal{C}_\infty) \longrightarrow (\text{Cok}_n(\mathcal{C}_\infty))_{\text{tor}}. \quad (4.11)$$

From (4.11), (4.10), and (4.9), we deduce that for  $n$  large enough, we have

$$0 \longrightarrow \text{I}(\mathcal{E}'_\infty/\mathcal{E}_\infty)/\text{Im}(\text{I}(\mathcal{E}'_\infty/\mathcal{C}_\infty)) \longrightarrow \text{Ker}_n(\mathcal{E}_\infty/\mathcal{C}_\infty) \longrightarrow (\mathcal{S}_M/\mathcal{S}')_{\text{tor}}. \quad (4.12)$$

We have a canonical injection  $\mathcal{E}'_\infty/\mathcal{E}_\infty \hookrightarrow \mathcal{U}_\infty/\mathcal{E}_\infty$ , and  $\mathcal{U}_\infty/\mathcal{E}_\infty$  is a quotient of  $\mathcal{U}_\infty/\mathcal{C}_\infty$ . By Lemma 4.2, we deduce that  $\text{char}_{\Lambda'}(\mathcal{E}'_\infty/\mathcal{E}_\infty)$  is prime to  $1 - \gamma_n$ , hence  $(\mathcal{E}'_\infty/\mathcal{E}_\infty)_{\Gamma_n}$  and  $(\mathcal{E}'_\infty/\mathcal{E}_\infty)^{\Gamma_n}$  are finite. For  $n$  large enough,  $\text{I}(\mathcal{E}'_\infty/\mathcal{E}_\infty) = (\mathcal{E}'_\infty/\mathcal{E}_\infty)^{\Gamma_n}$ , and the lemma follows from (4.12).  $\square$

**Lemma 4.4** *There exists  $M \in \mathbb{N}$ , such that  $N'_{K_n/D_n}(\mathcal{C}_n) = N'_{K_n/D_n}(\mathcal{C}_M \cap \mathcal{E}_{k(\mathfrak{f})})$  for all  $n \geq M$ .*

*Proof.* We set  $|z| := z\bar{z}$  for any  $z \in \mathbb{C}$ . For any finite abelian extension  $F/k$ , we denote by  $\ell_F$  the Dirichlet logarithm below,

$$\ell_F : F^\times \rightarrow \mathbb{C}[\text{Gal}(F/k)], \quad x \mapsto \sum_{\sigma \in \text{Gal}(F/k)} \log |x^\sigma| \sigma^{-1},$$

whose kernel verify  $\text{Ker}(\ell_F) \cap \mathcal{O}_F^\times = \mu(F)$ . From (2.2), we deduce

$$\ell_{k(\mathfrak{m})}(\Psi'(k(\mathfrak{m}), \mathfrak{m})) \subseteq \bigoplus_{\substack{\chi \in \text{Gal}(k(\mathfrak{m})/k) \\ \chi \neq 1}} \mathbb{C} e_\chi \ell_{k(\mathfrak{m})}(\varphi_{\mathfrak{m}}(1)), \quad (4.13)$$

for any nonzero proper ideal  $\mathfrak{m}$  of  $\mathcal{O}_k$ .

Let  $m \in \mathbb{N}^*$ , and let  $\mathfrak{g}$  be a nonzero ideal of  $\mathcal{O}_k$  which divides  $\mathfrak{f}$ . Let  $D'$  be the subfield of  $k(\mathfrak{gp}^m)$  fixed by the decomposition group of  $\mathfrak{p}$  in  $k(\mathfrak{gp}^m)$ , let  $\chi \neq 1$  be an irreducible complex character of  $\text{Gal}(D'/k)$ , and let  $\tilde{\chi}$  be the character on  $\text{Gal}(k(\mathfrak{gp}^m)/k)$  defined by  $\chi$ . By [9, Théorème 10],  $e_{\tilde{\chi}} \ell_{k(\mathfrak{gp}^m)}(\varphi_{\mathfrak{gp}^m}(1)) = 0$ . Then from (4.13) we deduce that

$$\ell_{k(\mathfrak{gp}^m)}\left(\Psi'(k(\mathfrak{gp}^m), \mathfrak{gp}^m)^{\text{Gal}(k(\mathfrak{gp}^m)/D')}\right) \subseteq \bigoplus_{\substack{\chi \in \text{Gal}(D'/k) \\ \chi \neq 1}} \mathbb{C} e_{\tilde{\chi}} \ell_{k(\mathfrak{gp}^m)}(\varphi_{\mathfrak{gp}^m}(1)) = 0. \quad (4.14)$$

Since  $k(\mathfrak{gp}^m) \cap D_n \subseteq D'$ , we deduce from (4.14) that  $\Psi'(D_n, \mathfrak{gp}^m) \subseteq \mu(D_n)$ . Then  $N_{K_n/D_n}(\tilde{\mathcal{C}}_n) \subseteq (\mathcal{U}_{D_n})_{\text{tor}}$  by Lemma 2.2. From Remark 2.3, we deduce that  $N'_{K_n/D_n}(\mathcal{C}_n)$  is generated by  $N'(\mathcal{C}_n \cap \mathcal{E}_{k(\mathfrak{f})})$ . Lemma 4.4 follows because  $\mathcal{E}_{k(\mathfrak{f})}$  is noetherian.  $\square$

**Lemma 4.5** *Set  $\delta := [D_\infty : k]$ . There is  $(\rho_1, \rho_2) \in (\mathbb{Q}_+^*)^2$  such that for all  $K_n \supseteq D_\infty$ ,*

$$\rho_1 p^{(\delta-1)n} \leq \# \left( \mathcal{U}_{D_\infty}^{(0)} / N_{K_n/D_\infty}(\mathcal{U}_n^{(0)}) \right) \leq \rho_2 p^{\delta n}.$$

*Proof.* Since  $N_{K_n/D_\infty}(\mathcal{U}_n^{(0)}) = N_{K_n/D_\infty}(\mathcal{U}_n) \cap \mathcal{U}_{D_\infty}^{(0)}$ , we have the exact sequence

$$0 \longrightarrow \frac{\mathcal{U}_{D_\infty}^{(0)}}{N_{K_n/D_\infty}(\mathcal{U}_n^{(0)})} \longrightarrow \frac{\mathcal{U}_{D_\infty}}{N_{K_n/D_\infty}(\mathcal{U}_n)} \longrightarrow \frac{\mathcal{U}_{D_\infty}}{\mathcal{U}_{D_\infty}^{(0)} N_{K_n/D_\infty}(\mathcal{U}_n)} \longrightarrow 0. \quad (4.15)$$

Let  $R_\infty$  be the maximal subextension of  $K_\infty$  which is unramified over  $\mathfrak{p}$ . Since the quotient  $\mathcal{U}_{D_\infty}^{(0)}/N_{K_n/D_\infty}(\mathcal{U}_n^{(0)})$  is finite for all  $n$  with  $D_\infty \subseteq K_n$ , we can assume that  $R_\infty \subseteq K_n$ . From class field theory, we have an isomorphism  $\frac{\mathcal{U}_{D_\infty}}{N_{K_n/D_\infty}(\mathcal{U}_n)} \simeq \prod_{\mathfrak{q}|\mathfrak{p}} I_{\mathfrak{q}}$ , where for all prime ideal  $\mathfrak{q}$  of  $\mathcal{O}_{K_n}$  above  $\mathfrak{p}$ ,  $I_{\mathfrak{q}}$  is the  $p$ -part of the inertia group of  $\mathfrak{q}$  in  $K_n/D_\infty$ , which is the  $p$ -part of  $\text{Gal}(K_n/R_\infty)$ . The number of primes  $\mathfrak{q}$  in  $D_\infty$  such that  $\mathfrak{q}|\mathfrak{p}$  is  $\delta$ , so if  $v_1$  is the valuation at  $p$  of  $[K_0 : k][R_\infty : k]^{-1}$ , we have

$$\# \left( \frac{\mathcal{U}_{D_\infty}}{N_{K_n/D_\infty}(\mathcal{U}_n)} \right) = p^{\delta(v_1+n)}. \quad (4.16)$$

The  $\mathbb{Z}_p$ -module  $\mathcal{U}_{D_\infty}/\mathcal{U}_{D_\infty}^{(0)}$  is free of rank 1, and so we have

$$\# \left( \frac{\mathcal{U}_{D_\infty}}{\mathcal{U}_{D_\infty}^{(0)} N_{K_n/D_\infty}(\mathcal{U}_n)} \right) \leq p^{v_2+n}, \quad (4.17)$$

where  $v_2$  is the valuation at  $p$  of  $[K_0 : k][D_\infty : k]^{-1}$ . Now Lemma 4.5 follows from (4.15), (4.16), and (4.17).  $\square$

By Proposition 3.1, we have  $\text{Cok}_n(\mathcal{U}_\infty^{(0)}) \simeq N'_{K_n/D_n}(\mathcal{U}_n^{(0)})$ , and the exact sequence

$$0 \longrightarrow N'_{K_n/D_n}(\mathcal{C}_n) \longrightarrow N'_{K_n/D_n}(\mathcal{U}_n^{(0)}) \longrightarrow \text{Cok}_n(\mathcal{U}_\infty^{(0)}/\mathcal{C}_\infty) \longrightarrow 0. \quad (4.18)$$

**Proposition 4.2** *The orders of  $\text{Cok}_n(\mathcal{U}_\infty^{(0)}/\mathcal{C}_\infty)$  and of  $\text{Cok}_n(\mathcal{E}_\infty/\mathcal{C}_\infty)$  are bounded.*

*Proof.* By Lemma 4.3, it is sufficient to show that the orders of  $\text{Cok}_n(\mathcal{U}_\infty^{(0)}/\mathcal{C}_\infty)$  are bounded. Let  $M \in \mathbb{N}$  be as in Lemma 4.4, large enough so that  $D_M = D_\infty$ , and set  $\mathcal{Z} := \mathcal{E}_{k(f)} \cap \mathcal{C}_M$ . From Lemma 4.4 and (4.18), it is sufficient to show that the orders of  $N_{K_n/D_\infty}(\mathcal{U}_n^{(0)})/N_{K_n/D_\infty}(\mathcal{Z})$  are bounded independantly of  $n \geq M$ .

By (4.1) and Lemma 4.4, we see that  $\mathcal{U}_{D_\infty}^{(0)}/N_{K_n/D_\infty}(\mathcal{Z})$  is finite. For any  $x \in \mathcal{Z}$ , we have  $N_{K_n/D_\infty}(x) = N_{K_M/D_\infty}(x)^{p^{n-M}}$ . We set

$$\rho' := p^{(1-\delta)M} \# \left( \mathcal{U}_{D_\infty}^{(0)}/N_{K_M/D_\infty}(\mathcal{Z}) \right) \# \left( \left( \mathcal{U}_{D_\infty}^{(0)} \right)_{\text{tor}} \right),$$

and then we have

$$\# \left( \mathcal{U}_{D_\infty}^{(0)}/N_{K_n/D_\infty}(\mathcal{Z}) \right) \leq \rho' p^{(\delta-1)n} \quad (4.19)$$

From (4.19) and Lemma 4.5, we deduce

$$\# \left( N_{K_n/D_\infty}(\mathcal{U}_n^{(0)})/N_{K_n/D_\infty}(\mathcal{Z}) \right) \leq \rho' \rho_1^{-1}.$$

$\square$

## 5 Global units modulo elliptic units versus the ideal class group.

**Theorem 5.1** *The  $\Lambda'$ -modules  $\mathcal{E}_\infty/\mathcal{C}_\infty$  and  $A_\infty$  share the same  $\lambda$ -invariant and the same  $\mu$ -invariant. The  $\Lambda'$ -modules  $\mathcal{U}_\infty/\mathcal{C}_\infty$  and  $B_\infty$  share the same  $\lambda$ -invariant and the same  $\mu$ -invariant.*



*Proof.* By (3.2), we have the exact sequence below,

$$0 \longrightarrow \mathcal{E}_\infty/\mathcal{C}_\infty \longrightarrow \mathcal{U}_\infty/\mathcal{C}_\infty \longrightarrow B_\infty \longrightarrow A_\infty \longrightarrow 0,$$

from which we deduce

$$\begin{cases} \lambda(\mathcal{E}_\infty/\mathcal{C}_\infty) \lambda(B_\infty) &= \lambda(\mathcal{U}_\infty/\mathcal{C}_\infty) \lambda(A_\infty), \\ \mu(\mathcal{E}_\infty/\mathcal{C}_\infty) \mu(B_\infty) &= \mu(\mathcal{U}_\infty/\mathcal{C}_\infty) \mu(A_\infty). \end{cases} \quad (5.1)$$

By (5.1), we just have to show the first assertion of the theorem. For any two sequences  $(u_n)_{n \in \mathbb{N}}$  and  $(v_n)_{n \in \mathbb{N}}$  in  $\mathbb{N}^*$ , let us write  $u_n \sim v_n$  if  $(u_n)_{n \in \mathbb{N}}$  and  $(v_n)_{n \in \mathbb{N}}$  are asymptotically equivalent. Then by Iwasawa's theorem and [7, Théorème p300, and discussion p301], we have

$$p^{\mu(A_\infty)p^n + \lambda(A_\infty)n} \sim \#(A_n) \sim \#(\mathcal{E}_n/\mathcal{C}_n^R). \quad (5.2)$$

Applying Lemma 2.3 to (5.2), and then by Proposition 4.1 and Proposition 4.2, we have

$$p^{\mu(A_\infty)p^n + \lambda(A_\infty)n} \sim \#(\mathcal{E}_n/\mathcal{C}_n) \sim \#(\mathcal{E}_\infty/\mathcal{C}_\infty)_{\Gamma_n}. \quad (5.3)$$

By the general theory of  $\Lambda'$ -modules, we deduce from (5.3) that

$$p^{\mu(A_\infty)p^n + \lambda(A_\infty)n} \sim p^{\mu(\mathcal{E}_\infty/\mathcal{C}_\infty)p^n + \lambda(\mathcal{E}_\infty/\mathcal{C}_\infty)n},$$

from which the theorem follows. □

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